MASSACHUSETTS CURRICULUM FRAMEWORK FOR MATHEMATICS

Grades Pre-Kindergarten to 12

Incorporating the
Common Core State Standards for Mathematics

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Massachusetts Department of
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The Standards for Mathematical Content

HIGH SCHOOL: CONCEPTUAL CATEGORIES

The high school content standards specify the mathematics that all students should study in order to be college and career ready. Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by a (+) symbol, as in this example:

N-CN.4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers).

All standards without a (+) symbol should be in the common mathematics curriculum for all college and career ready students. Standards with a (+) symbol may also appear in courses intended for all students.

Organization of Standards

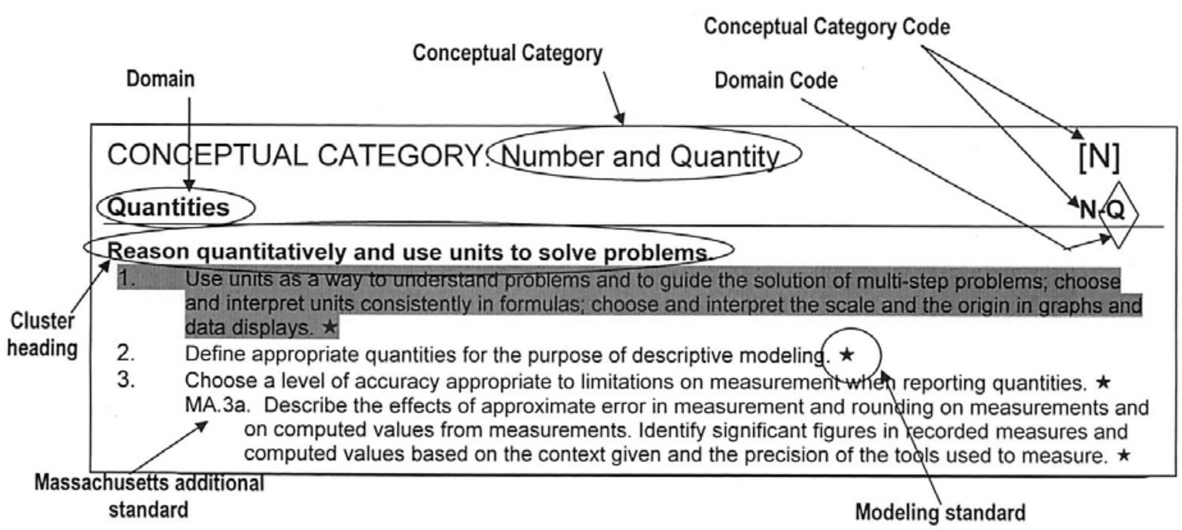
The high school standards are listed in **conceptual categories**:

- Number and Quantity (N)
- Algebra (A)
- Functions (F)
- Modeling (★)
- Geometry (G)
- Statistics and Probability (S)

Conceptual categories portray a coherent view of high school mathematics; a student's work with functions, for example, crosses a number of traditional course boundaries, potentially up through and including calculus. Similar to the grade level content standards, each conceptual category (except Modeling, see explanation following the illustration) is further subdivided into several domains, and each domain is subdivided into clusters.

Standards Identifiers/Coding

High school content standards are identified first by conceptual category, rather than by grade as for pre-kindergarten through grade 8 content standards. The code for each high school standard begins with the identifier for the conceptual category code (N, A, F, G, S), followed by the domain code, and the standard number, as shown below.



The standard highlighted above is identified as N-Q.1, identifying it as a standard in the Number and Quantity conceptual category ("N-") within that category's Quantities domain ("Q"), and as the first standard in that domain.

The star symbol (★) following the standards in the illustration indicates those are also Modeling standards. Modeling is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Unique Massachusetts Standards

High school content standards unique to Massachusetts are initially coded with “MA.” In the illustration on the previous page, the Massachusetts addition “Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure. ★” is identified as MA.N-Q.3.a., indicating that it is a Massachusetts addition (“MA”) to the Number and Quantity conceptual category (“N-”) in the Quantities domain (“Q”), and that it is further specification to the N-Q.3 standard.

Introduction

Numbers and Number Systems

During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3…. Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers. (See Illustration 1 in the Glossary.)

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings.

Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities

In real-world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

CONCEPTUAL CATEGORY: Number and Quantity [N]

Overview

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

Quantities

- Reason quantitatively and use units to solve problems.

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.
- Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Content Standards

The Real Number System N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities N-Q

Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★
   MA.3.a. Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure. ★

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, $\left(-1+\sqrt{3}i\right)^3 = 8$ because $\left(-1+\sqrt{3}i\right)$ has modulus 2 and argument 120°.*
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations.

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.*
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Vector and Matrix Quantities N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.
   a. (+) Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that (+) the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. (+) Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. (+) Understand vector subtraction $\boldsymbol{v} - \boldsymbol{w}$ as $\boldsymbol{v} + (-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
   a. (+) Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
   b. (+) Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

(+) indicates standard beyond College and Career Ready.

Introduction

Expressions

An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Equations and Inequalities

An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 – 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1+b_2)/2)h$, can be solved for h using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Connections to Functions and Modeling

Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

Overview

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Content Standards

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
   c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*
4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
   MA.1.a. Divide polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[32]

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
32 The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

Creating Equations — A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.* ★

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning.

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

   MA.3.a. Solve linear equations and inequalities in one variable involving absolute value.
4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

   MA.4.c. Demonstrate an understanding of the equivalence of factoring, completing the square, or using the quadratic formula to solve quadratic equations.

Solve systems of equations.

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.*
8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Represent and solve equations and inequalities graphically.

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

★ indicates Modeling standard.

Introduction

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city"; by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates

Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

Overview

Interpreting Functions

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Content Standards

Interpreting Functions F-IF

Understand the concept of a function and use function notation.

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, *and* $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*
   MA.8.c. Translate among different representations of functions and relations: graphs, equations, point sets, and tables.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

MA.10. Given algebraic, numeric and/or graphical representations of functions, recognize the function as polynomial, rational, logarithmic, exponential, or trigonometric.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

CONCEPTUAL CATEGORY: Functions [F]

Building Functions F-BF

Build a function that models a relationship between two quantities.

1. Write a function that describes a relationship between two quantities.★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
   c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x - 1)$ for $x \neq 1$.*
   b. (+) Verify by composition that one function is the inverse of another.
   c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
   d. (+) Produce an invertible function from a non-invertible function by restricting the domain.
5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★
4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★

Interpret expressions for functions in terms of the situation they model.

5. Interpret the parameters in a linear or exponential function in terms of a context. ★

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $^{\pi}/_{3}$, $^{\pi}/_{4}$ and $^{\pi}/_{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★
6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.
9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Introduction

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

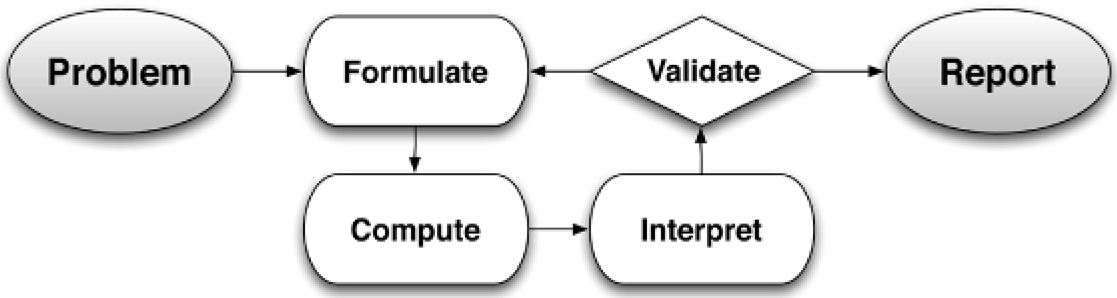
Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram below. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific Modeling standards appear throughout the high school standards indicated by a star symbol (★).

Introduction

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college, some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of “same shape” and “scale factor” developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations

The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

CONCEPTUAL CATEGORY: Geometry [G]

Overview

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Prove geometric theorems.
- Make geometric constructions.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles.
- Apply trigonometry to general triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.
- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

STANDARDS FOR MATHEMATICAL PRACTICE
1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for an express regularity in repeated reasoning.

Content Standards

Congruence **G-CO**

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions.

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems.

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*
    MA.11.a. Prove theorems about polygons. *Theorems include: measures of interior and exterior angles, properties of inscribed polygons.*

Make geometric constructions.

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

Prove theorems involving similarity.

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
   MA.3.a. Derive the formula for the relationship between the number of sides and sums of the interior and sums of the exterior angles of polygons and apply to the solutions of mathematical and contextual problems.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles.

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.
3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
   MA.3.a. (+) Use equations and graphs of conic sections to model real-world problems. ★

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,* $\sqrt{3}$*) lies on the circle centered at the origin and containing the point (0, 2).*
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

Geometric Measurement and Dimension — G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry — G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

MA.4. Use dimensional analysis for unit conversions to confirm that expressions and equations make sense. ★

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Introduction

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

Connections to Functions and Modeling

Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.

Overview

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments and observational studies.

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Calculate expected values and use them to solve problems.
- Use probability to evaluate outcomes of decisions.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Content Standards

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Summarize, represent, and interpret data on two categorical and quantitative variables.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

★ indicates Modeling standard.

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data.

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not”). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*’s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Using Probability to Make Decisions S-MD

Calculate expected values and use them to solve problems.

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. ★
2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. ★
3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.* ★
4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?* ★

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Use probability to evaluate outcomes of decisions.

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.★
   a. (+) Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* ★
   b. (+) Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.* ★
6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).[33] ★

[33] Replacing the hockey goalie with an extra skater.

★ indicates Modeling standard.

(+) indicates standard beyond College and Career Ready.

The Standards for Mathematical Content

HIGH SCHOOL: MODEL PATHWAYS AND MODEL COURSES

The Progression from Grade 8 Standards to Model Algebra I or Model Mathematics I Standards

The pre-kindergarten to grade 8 standards present a coherent progression of concepts and skills that will prepare students for the Model Traditional Pathway's Model Algebra I course or the Model Integrated Pathway's Model Mathematics I course. Students will need to master the grades 6–8 standards in order to be prepared for the Model Algebra I course or Model Mathematics I course presented in this document. Some students may master the 2011 grade 8 standards earlier than grade 8, which would enable these students to take the high school Model Algebra I course or Model Mathematics I course in grade 8.

The 2011 grade 8 standards are rigorous; students are expected to learn about linear relationships and equations to begin the study of functions and compare rational and irrational numbers. In addition, the statistics presented in the grade 8 standards are more sophisticated and include connecting linear relations with the representation of bivariate data. The Model Algebra I and Model Mathematics I courses progress from these concepts and skills, and focus on quadratic and exponential functions. Thus, the 2011 Model Algebra I course is a more advanced course than the Algebra I course identified in the 2000 *Massachusetts Curriculum Framework for Mathematics*. Likewise, the Model Mathematics I course is also designed to follow the more rigorous 2011 grade 8 standards.

Development of High School Model Pathways and Model Courses[34]

The 2011 grades 9–12 high school mathematics standards presented by conceptual categories provide guidance on what students are expected to learn in order to be prepared for college and careers. When presented by conceptual categories, these standards do not indicate a sequence of high school courses. Massachusetts educators requested additional guidance about how these 9–12 standards might be configured into model high school courses and represent a smooth transition from the grades pre-k–8 standards.

Achieve (in partnership with the Common Core writing team) convened a group of experts, including state mathematics experts, teachers, mathematics faculty from two- and four-year institutions, mathematics teacher educators, and workforce representatives, to develop model course pathways in mathematics based on the high school conceptual category standards in the Common Core State Standards. Two Model Pathways of model courses, Traditional (Algebra I, Geometry, Algebra II) and Integrated (Mathematics I, Mathematics II, Mathematics III), resulted and were originally presented in the June 2010 *Common Core State Standards for Mathematics Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards for Mathematics*.

The Massachusetts Department of Elementary and Secondary Education convened high school teachers, higher education faculty, and business leaders to review the two Model Pathways and related model courses, and to create two additional model advanced courses that students may choose to take after completing either Model Pathway. The Model Pathways and model courses included in this *Framework* are adapted from those in *Common Core State Standards for Mathematics Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards for Mathematics*.

[34] Adapted from the *Common Core State Standards for Mathematics* and *Appendix A: Designing High School Courses based on the Common Core State Standards for Mathematics.*

Introduction:

High School Content Standards/Model Pathways and Model Courses

The Model Pathways and Model Courses

The following Model Pathways and model courses are presented in this *Framework*:

- Model Traditional Pathway
  - Model Algebra I
  - Model Geometry
  - Model Algebra II
- Model Integrated Pathway
  - Model Mathematics I
  - Model Mathematics II
  - Model Mathematics III
- Advanced Model Courses
  - Model Precalculus
  - Model Advanced Quantitative Reasoning

All of the College and Career Ready high school content standards presented by conceptual categories[35] are included in appropriate locations within the three model courses of both Model Pathways. Students completing either Model Pathway are prepared for additional courses, such as the model advanced courses that follow the Model Pathways. Model advanced courses are comprised of the higher-level mathematics standards (+) in the conceptual categories.

The Model Traditional Pathway reflects the approach typically seen in the U.S., consisting of two model algebra courses with some Statistics and Probability standards included, and a model geometry course, with some Number and Quantity standards and some Statistics and Probability standards included. The Model Integrated Pathway reflects the approach typically seen internationally, consisting of a sequence of three model courses, each of which includes Number and Quantity, Algebra, Functions, Geometry, and Statistics and Probability standards.

While the Model Pathways and model courses organize the Standards for Mathematical Content into model pathways to college and career readiness, the content standards must also be connected to the Standards for Mathematical Practice to ensure that the students increasingly engage with the subject matter as they grow in mathematical maturity and expertise.

Organization of the Model High School Courses

Each model high school course is presented in three sections:

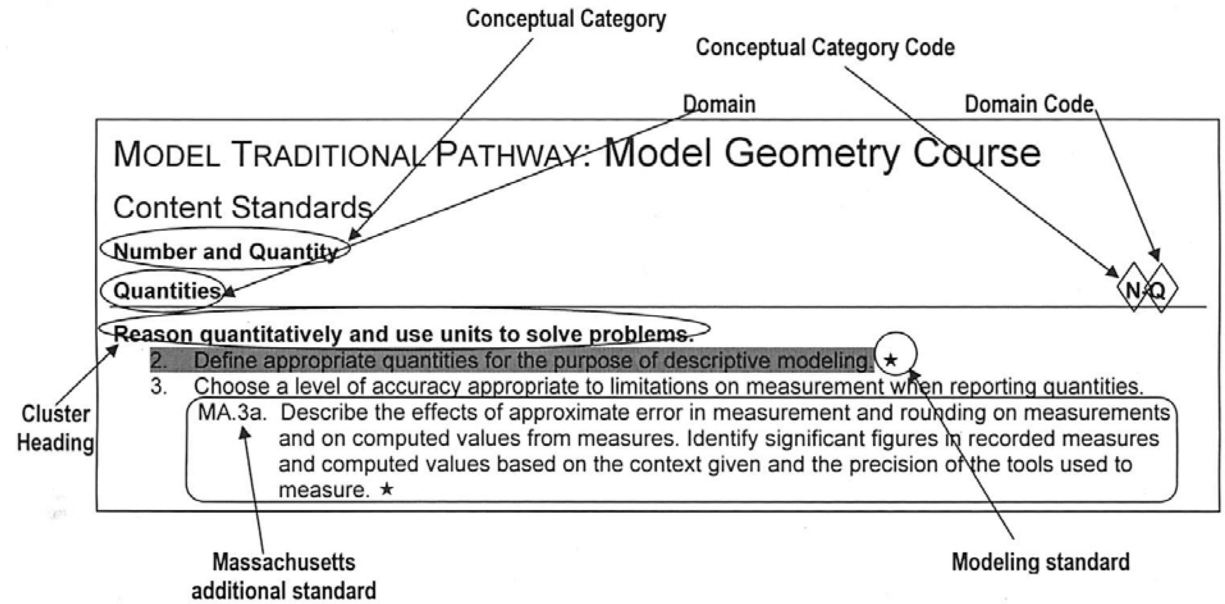
- an introduction and description of the critical areas for learning in that course;
- an overview listing the conceptual categories, domains, and clusters included in that course; and
- the content standards for that course, presented by conceptual category, domain, and cluster.

Standards Identifiers/Coding

Standard numbering in the high school model courses is identical to the coding presented in the introduction to the high school standards by conceptual category.

The illustration on the following page shows a section from the Model Geometry course content standards. The standard highlighted in the illustration is standard N-Q.2, identifying it as a standard from the Number and Quantity conceptual category (“N-”), in the Quantity domain (“Q”), and as the second standard in that domain. The star (★) at the end of the standard indicates that it is a Modeling standard. Note that standard N-Q.1 from the Number and Quantity conceptual category is not included in the Model Geometry course; N-Q.1 is included in the Model Algebra I course.

[35] In select cases (+) standards are included in Model Pathway model courses to maintain mathematical coherence.



As in the conceptual category presentation of the content standards, a plus sign (+) at the beginning of a standard indicates higher-level mathematics skills and knowledge that students should learn in order to take more advanced mathematics courses such as Calculus, and the star symbol (★) at the end of a standard indicates a Modeling standard (see below).

Importance of Modeling in High School

Modeling (indicated by a ★ at the end of a standard) is defined as both a *conceptual category* for high school mathematics and a *Standard for Mathematical Practice*, and is an important avenue for motivating students to study mathematics, for building their understanding of mathematics, and for preparing them for future success. Development of the Model Pathways into instructional programs will require careful attention to modeling and the mathematical practices. Assessments based on these Model Pathways should reflect both the Standards for Mathematical Content and the Standards for Mathematical Practice.

Footnotes for Repeated Standards

It is important to note that some standards are repeated in two or more model courses within a Model Pathway. Footnotes for these standards clarify the aspect(s) of the duplicated standard relevant to each model course; these footnotes are an important part of the standards for each model course.

For example, the following standard is included in both the Model Algebra I course and the Model Algebra II course, with the appropriate footnotes in each model course:

> A-APR.1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

The footnote in Model Algebra I, “For Algebra I, focus on adding and multiplying polynomial expressions, factoring or expanding expressions to identify and collect like terms, applying the distributive property,” indicates that operations with polynomials is limited in Model Algebra I.

The same standard in Model Algebra II does not have a footnote, indicating that the standard has no limitations in Model Algebra II.

Introduction

The fundamental purpose of the Model Algebra I course is to formalize and extend the mathematics that students learned in the middle grades. This course is comprised of standards selected from the high school **conceptual categories**, which were written to encompass the scope of content and skills to be addressed throughout grades 9–12 rather than through any single course. Therefore, the complete standard is presented in the model course, with clarifying footnotes as needed to limit the scope of the standard and indicate what is appropriate for study in this particular course. For example, the scope of Model Algebra I is limited to linear, quadratic, and exponential expressions and functions as well as some work with absolute value, step, and functions that are piecewise-defined. Therefore, although a standard may include references to logarithms or trigonometry, those functions are not to be included in coursework for Model Algebra I; they will be addressed later in Model Algebra II. Reminders of this limitation are included as footnotes where appropriate in the Model Algebra I standards.

For the high school Model Algebra I course,[36] instructional time should focus on four critical areas: (1) deepen and extend understanding of linear and exponential relationships; (2) contrast linear and exponential relationships with each other and engage in methods for analyzing, solving, and using quadratic functions; (3) extend the laws of exponents to square and cube roots; and (4) apply linear models to data that exhibit a linear trend.

(1) By the end of eighth grade, students have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. In Algebra I, students analyze and explain the process of solving an equation and justify the process used in solving a system of equations. Students develop fluency writing, interpreting, and translating among various forms of linear equations and inequalities, and use them to solve problems. They master the solution of linear equations and apply related solution techniques and the laws of exponents to the creation and solution of simple exponential equations.

(2) In earlier grades, students define, evaluate, and compare functions, and use them to model relationships between quantities. In Algebra I, students learn function notation and develop the concepts of domain and range. They focus on linear, quadratic, and exponential functions, including sequences, and also explore absolute value, step, and piecewise-defined functions; they interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations. Students build on and extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. Students explore systems of equations and inequalities, and they find and interpret their solutions. They interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

(3) Students extend the laws of exponents to rational exponents involving square and cube roots and apply this new understanding of number; they strengthen their ability to see structure in and create quadratic and exponential expressions. They create and solve equations, inequalities, and systems of equations involving quadratic expressions. Students become facile with algebraic manipulation, including rearranging and collecting terms, and factoring, identifying, and canceling common factors in rational expressions. Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from among these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. Students expand their experience with functions to include more specialized functions—absolute value, step, and those that are piecewise-defined.

[36] Adapted from the *Common Core State Standards for Mathematics* and *Appendix A: Designing High School Courses based on the Common Core State Standards for Mathematics.*

(4) Building upon their prior experiences with data, students explore a more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

MODEL TRADITIONAL PATHWAY: Model Algebra I

Overview

Number and Quantity

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

Quantities

- Reason quantitatively and use units to solve problems.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Functions (cont'd.)

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

Content Standards

Number and Quantity

The Real Number System — N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities — N-Q

Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★
   MA.3.a. Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure. ★

Algebra

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions.[37]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
   c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

[37] Algebra I is limited to linear, quadratic, and exponential expressions.
★ indicates Modeling standard.

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.[38]

Creating Equations[39] — A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities,[40] and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.* ★

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning.

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

   MA.3.a. Solve linear equations and inequalities in one variable involving absolute value.

4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions[41] and write them as $a \pm bi$ for real numbers a and b.

   MA.4.c. Demonstrate an understanding of the equivalence of factoring, completing the square, or using the quadratic formula to solve quadratic equations.

Solve systems of equations.

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

[38] For Algebra I, focus on adding and multiplying polynomial expressions, factoring or expanding polynomial expressions to identify and collect like terms, applying the distributive property.
[39] Create linear, quadratic, and exponential (with integer domain) equations in Algebra I.
★ indicates Modeling standard.
[40] Equations and inequalities in this standard should be limited to linear.
[41] It is sufficient in Algebra I to recognize when roots are not real; writing complex roots is included in Algebra II.

7. Solve a simple system consisting of a linear equation and a quadratic[42] equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*

Represent and solve equations and inequalities[43] graphically.

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions F-IF

Understand the concept of a function and use function notation.

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ for $n \geq 1$.*

Interpret functions[44] that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

[42] Algebra I does not include the study of conic equations; include quadratic equations typically included in Algebra I.
[43] In Algebra I, functions are limited to linear, absolute value, and exponential functions for this cluster.
[44] Limit to interpreting linear, quadratic, and exponential functions.
★ indicates Modeling standard.

Analyze functions[45] using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root,[46] and piecewise-defined functions, including step functions and absolute value functions. ★
   e. Graph exponential and logarithmic[47] functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.[48] ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, and $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.*
   MA.8.c. Translate among different representations of functions and relations: graphs, equations, point sets, and tables.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

MA.10. Given algebraic, numeric and/or graphical representations of functions, recognize the function as polynomial, rational, logarithmic, exponential, or trigonometric.

Building Functions[49] F-BF

Build a function that models a relationship between two quantities.

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula,[50] use them to model situations, and translate between the two forms. ★

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x - 1)$ for $x \neq 1$.*

[45] In Algebra I, only linear, exponential, quadratic, absolute value, step, and piecewise functions are included in this cluster.
★ indicates Modeling standard.
[46] Graphing square root and cube root functions is included in Algebra II.
[47] In Algebra I it is sufficient to graph exponential functions showing intercepts.
[48] Showing end behavior of exponential functions and graphing logarithmic and trigonometric functions is not part of Algebra I.
[49] Functions are limited to linear, quadratic, and exponential in Algebra I.
[50] In Algebra I, identify linear and exponential sequences that are defined recursively; continue the study of sequences in Algebra II.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
    a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
    b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
    c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.

5. Interpret the parameters in a linear or exponential[51] function in terms of a context. ★

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★[52]

Summarize, represent, and interpret data on two categorical and quantitative variables.[53]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
    a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
    b. Informally assess the fit of a function by plotting and analyzing residuals. ★
    c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

★ indicates Modeling standard.

[51] Limit exponential function to the form $f(x) = b^x + k$.

[52] Introduce in Algebra I; expand in Algebra II.

[53] Linear focus; discuss as a general principle in Algebra I.

Introduction

The fundamental purpose of the Model Geometry course is to formalize and extend students' geometric experiences from the middle grades. This course is comprised of standards selected from the high school **conceptual categories**, which were written to encompass the scope of content and skills to be addressed throughout grades 9–12 rather than through any single course. Therefore, the complete standard is presented in the model course, with clarifying footnotes as needed to limit the scope of the standard and indicate what is appropriate for study in this particular course.

In this high school Model Geometry course,[54] students explore more complex geometric situations and deepen their explanations of geometric relationships, presenting and hearing formal mathematical arguments. Important differences exist between this course and the historical approach taken in geometry classes. For example, transformations are emphasized in this course. Close attention should be paid to the introductory content for the Geometry conceptual category found on page 92.

For the high school Model Geometry course, instructional time should focus on six critical areas: (1) establish criteria for congruence of triangles based on rigid motions; (2) establish criteria for similarity of triangles based on dilations and proportional reasoning; (3) informally develop explanations of circumference, area, and volume formulas; (4) apply the Pythagorean Theorem to the coordinate plan; (5) prove basic geometric theorems; and (6) extend work with probability.

(1) Students have prior experience with drawing triangles based on given measurements and performing rigid motions including translations, reflections, and rotations. They have used these to develop notions about what it means for two objects to be congruent. In this course, students establish triangle congruence criteria, based on analyses of rigid motions and formal constructions. They use triangle congruence as a familiar foundation for the development of formal proof. Students prove theorems—using a variety of formats including deductive and inductive reasoning and proof by contradiction—and solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why they work.

(2) Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. Students derive the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles, building on their work with quadratic equations done in Model Algebra I. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles.

(3) Students' experience with three-dimensional objects is extended to include informal explanations of circumference, area, and volume formulas. Additionally, students apply their knowledge of two-dimensional shapes to consider the shapes of cross-sections and the result of rotating a two-dimensional object about a line.

(4) Building on their work with the Pythagorean Theorem in eighth grade to find distances, students use the rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals, and slopes of parallel and perpendicular lines, which relates back to work done in the Model Algebra I course. Students continue their study of quadratics by connecting the geometric and algebraic definitions of the parabola.

[54] Adapted from the *Common Core State Standards for Mathematics* and *Appendix A: Designing High School Courses based on the Common Core State Standards for Mathematics*

(5) Students prove basic theorems about circles, with particular attention to perpendicularity and inscribed angles, in order to see symmetry in circles and as an application of triangle congruence criteria. They study relationships among segments on chords, secants, and tangents as an application of similarity. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center. Given an equation of a circle, they draw the graph in the coordinate plane, and apply techniques for solving quadratic equations—which relates back to work done in the Model Algebra I course—to determine intersections between lines and circles or parabolas and between two circles.

(6) Building on probability concepts that began in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

MODEL TRADITIONAL PATHWAY: Model Geometry

Overview

Number and Quantity

Quantities

- Reason quantitatively and use units to solve problems.

Geometry

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Prove geometric theorems.
- Make geometric constructions.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles.
- Apply trigonometry to general triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and area of sectors of circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.
- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Statistics and Probability

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

Content Standards

Number and Quantity

Quantities — N-Q

Reason quantitatively and use units to solve problems.

2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★
   MA.3.a. Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure. ★

Geometry

Congruence — G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions.

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems.[55]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

★ indicates a modeling standard.
[55] Proving the converse of theorems should be included when appropriate.

MA.11.a. Prove theorems about polygons. *Theorems include: measures of interior and exterior angles, properties of inscribed polygons.*

Make geometric constructions.

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

Prove theorems involving similarity.

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

MA.3.a. Derive the formula for the relationship between the number of sides and sums of the interior and sums of the exterior angles of polygons and apply to the solutions of mathematical and contextual problems.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles.

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically*. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,* $\sqrt{3}$*) lies on the circle centered at the origin and containing the point (0, 2).*
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
3. Use volume formulas[56] for cylinders, pyramids, cones, and spheres to solve problems. ★

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

MA.4. Use dimensional analysis for unit conversions to confirm that expressions and equations make sense. ★

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
56 Note: The 2011 Massachusetts grade 8 mathematics standards require that students know volume formulas for cylinders, cones, and spheres.

Statistics and Probability

Conditional Probability and the Rules of Probability — S-CP

Understand independence and conditional probability and use them to interpret data.[57]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not”). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as $P(A \text{ and } B)/P(B)$, and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.[58]

6. Find the conditional probability of *A* given *B* as the fraction of *B*’s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Using Probability to Make Decisions — S-MD

Use probability to evaluate outcomes of decisions.

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).[59] ★

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57 Link to data from simulations or experiments.
★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
58 Introductory only.
59 Replacing the hockey goalie with an extra skater.

Introduction

Building on their work with linear, quadratic, and exponential functions, students extend their repertoire of functions to include logarithmic, polynomial, rational, and radical functions in the Model Algebra II course. This course is comprised of standards selected from the high school **conceptual categories**, which were written to encompass the scope of content and skills to be addressed throughout grades 9–12 rather than through any single course. Therefore, the complete standard is presented in the model course, with clarifying footnotes as needed to limit the scope of the standard and indicate what is appropriate for study in this particular course. Standards that were limited in Model Algebra I no longer have those restrictions in Model Algebra II. Students work closely with the expressions that define the functions, are facile with algebraic manipulations of expressions, and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms.

For the high school Model Algebra II course,[60] instructional time should focus on four critical areas: (1) relate arithmetic of rational expressions to arithmetic of rational numbers; (2) expand understandings of functions and graphing to include trigonometric functions; (3) synthesize and generalize functions and extend understanding of exponential functions to logarithmic functions; and (4) relate data display and summary statistics to probability and explore a variety of data collection methods.

(1) A central theme of this Model Algebra II course is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students explore the structural similarities between the system of polynomials and the system of integers. They draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Connections are made between multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations. The Fundamental Theorem of Algebra is examined.

(2) Building on their previous work with functions and on their work with trigonometric ratios and circles in the Model Geometry course, students now use the coordinate plane to extend trigonometry to model periodic phenomena.

(3) Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying function. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as "*the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions*" is at the heart of this Model Algebra II course. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

(4) Students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

[60] Adapted from the *Common Core State Standards for Mathematics* and *Appendix A: Designing High School Courses based on the Common Core State Standards for Mathematics.*

MODEL TRADITIONAL PATHWAY: Model Algebra II

Overview

Number and Quantity

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on matrices and use matrices in applications.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Functions (cont'd.)

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent and interpret data on a single count or measurement variable.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments and observational studies.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

Number and Quantity

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x – 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\mathbf{v}$, $|\mathbf{v}|$, $||\mathbf{v}||$, v).
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
12. (+) Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 – y^4$ *as* $(x^2)^2 – (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 – y^2)(x^2 + y^2)$.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
   MA.1.a. Divide polynomials.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[61]

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations — A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning.

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically.

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

[61] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.
★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   - c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   - MA.8.c. Translate among different representations of functions and relations: graphs, equations, point sets, and tables.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

Build a function that models a relationship between two quantities.

1. Write a function that describes a relationship between two quantities. ★
   - b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   - a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) =2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★

★ indicates Modeling standard.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions.

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).[62] ★

★ indicates Modeling standard.
[62] Replacing the hockey goalie with an extra skater.
(+) indicates standard beyond College and Career Ready.

Introduction

The fundamental purpose of the Model Mathematics I course is to formalize and extend the mathematics that students learned in the middle grades. This course is comprised of standards selected from the high school **conceptual categories**, which were written to encompass the scope of content and skills to be addressed throughout grades 9–12 rather than through any single course. Therefore, the complete standard is presented in the model course, with clarifying footnotes as needed to limit the scope of the standard and indicate what is appropriate for study in this particular course. For example, the scope of Model Mathematics I is limited to linear and exponential expressions and functions as well as some work with absolute value, step, and functions that are piecewise-defined. Therefore, although a standard may include references to quadratic, logarithmic, or trigonometric functions, those functions should not be included in coursework for Model Mathematics I; they will be addressed in Model Mathematics II or III.

For the high school Model Mathematics I course,[63] instructional time should focus on six critical areas, each of which is described in more detail below: (1) extend understanding of numerical manipulation to algebraic manipulation; (2) synthesize understanding of function; (3) deepen and extend understanding of linear relationships; (4) apply linear models to data that exhibit a linear trend; (5) establish criteria for congruence based on rigid motions; and (6) apply the Pythagorean Theorem to the coordinate plane.

(1) By the end of eighth grade students have had a variety of experiences working with expressions and creating equations. Students become facile with algebraic manipulation in much the same way that they are facile with numerical manipulation. Algebraic facility includes rearranging and collecting terms, factoring, identifying and canceling common factors in rational expressions, and applying properties of exponents. Students continue this work by using quantities to model and analyze situations, to interpret expressions, and to create equations to describe situations.

(2) In earlier grades, students define, evaluate, and compare functions, and use them to model relationships among quantities. Students will learn function notation and develop the concepts of domain and range. They move beyond viewing functions as processes that take inputs and yield outputs and start viewing functions as objects in their own right. They explore many examples of functions, including sequences; interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations. They work with functions given by graphs and tables, keeping in mind that, depending upon the context, these representations are likely to be approximate and incomplete. Their work includes functions that can be described or approximated by formulas as well as those that cannot. When functions describe relationships between quantities arising from a context, students reason with the units in which those quantities are measured. Students build on and informally extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

(3) By the end of eighth grade, students have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Building on these earlier experiences, students analyze and explain the process of solving an equation, and justify the process used in solving a system of equations. Students develop fluency writing, interpreting, and translating among various forms of linear equations and inequalities, and use them to solve problems. They master the solution of linear equations and apply related solution techniques and the laws of exponents to the creation and solution of simple exponential equations. Students explore systems of equations and inequalities, and they find and interpret their solutions. All of this work is grounded on understanding quantities and on relationships among them.

[63] Adapted from the *Common Core State Standards for Mathematics* and *Appendix A: Designing High School Courses based on the Common Core State Standards for Mathematics*

(4) Students' prior experiences with data are the basis for the more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships among quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

(5) In previous grades, students were asked to draw triangles based on given measurements. They also have prior experience with rigid motions: translations, reflections, and rotations, and have used these to develop notions about what it means for two objects to be congruent. Students establish triangle congruence criteria, based on analyses of rigid motions and formal constructions. They solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why they work.

(6) Building on their work with the Pythagorean Theorem in eighth grade to find distances, students use a rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals and slopes of parallel and perpendicular lines.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Overview

Number and Quantity

Quantities

- Reason quantitatively and use units to solve problems.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Geometry

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Make geometric constructions.

Expressing Geometric Properties with Equations

- Use coordinates to prove simple geometric theorems algebraically.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

MODEL INTEGRATED PATHWAY: Model Mathematics I

Content Standards

Number and Quantity

Quantities[64] **N-Q**

Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★
   MA.3.a. Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure. ★

Algebra

Seeing Structure in Expressions[65] **A-SSE**

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

Creating Equations[66] **A-CED**

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*[67] ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.* ★

Reasoning with Equations and Inequalities **A-REI**

Understand solving equations as a process of reasoning and explain the reasoning.

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.[68]

Solve equations and inequalities in one variable.[69]

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
   MA.3.a. Solve linear equations and inequalities in one variable involving absolute value.

[64] Foundation for work with expressions, equations, and functions.
[65] Limit Mathematics I to linear expressions and exponential expressions with integer exponents.
★ indicates Modeling standard.
[66] Limit Mathematics I to linear and exponential equations with integer exponents.
[67] Limit to linear equations and inequalities.
[68] Master for linear equations and inequalities; learn as general principle to be expanded in Mathematics II and III.
[69] Limit Mathematics I to linear inequalities and exponential of a form $2^x = 1/16$.

Solve systems of equations.[70]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Represent and solve equations and inequalities graphically.[71]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions — F-IF

Understand the concept of a function and use function notation.[72]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context.[73]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations.[74]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

[70] Limit Mathematics I to systems of linear equations.
[71] Limit Mathematics I to linear and exponential equations; learn as general principle to be expanded in Mathematics II and III.
★ indicates Modeling standard.
[72] Focus on linear and exponential functions with integer domains and on arithmetic and geometric sequences.
[73] Focus on linear and exponential functions with integer domains.
[74] Limit Mathematics I to linear and exponential functions with integer domains.

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

MA.10. Given algebraic, numeric and/or graphical representations of functions, recognize the function as polynomial, rational, logarithmic, exponential, or trigonometric.

Building Functions F-BF

Build a function that models a relationship between two quantities.[75]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions.[76]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.[77]

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.[78]

5. Interpret the parameters in a linear or exponential function in terms of a context. ★

Geometry

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

[75] Limit Mathematics I to linear and exponential functions with integer domains.
★ indicates Modeling standard.
[76] Limit Mathematics I to linear and exponential functions; focus on vertical translations for exponential functions.
[77] Limit Mathematics I to linear and exponential models.
[78] Limit Mathematics I to linear and exponential functions of the form $f(x) – b^x + k$.

3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions.[79]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Make geometric constructions.[80]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Expressing Geometric Properties with Equations — G-GPE

Use coordinates to prove simple geometric theorems algebraically.[81]

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).*
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

Statistics and Probability

Interpreting Categorical and Quantitative Data — S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

[79] Build on rigid motions as a familiar starting point for development of geometric proof.
[80] Formalize proof, and focus on explanation of process.
[81] Include the distance formula and relate to the Pythagorean Theorem.
★ indicates Modeling standard.

Summarize, represent, and interpret data on two categorical and quantitative variables.[82]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

[82] Focus on linear applications; learn as general principle to be expanded in Mathematics II and III.

★ indicates Modeling standard.

Introduction

The focus of the Model Mathematics II course is on quadratic expressions, equations, and functions; comparing their characteristics and behavior to those of linear and exponential relationships from Model Mathematics I. This course is comprised of standards selected from the high school **conceptual categories**, which were written to encompass the scope of content and skills to be addressed throughout grades 9–12 rather than through any single course. Therefore, the complete standard is presented in the model course, with clarifying footnotes as needed to limit the scope of the standard and indicate what is appropriate for study in this particular course. For example, the scope of Model Mathematics II is limited to quadratic expressions and functions, and some work with absolute value, step, and functions that are piecewise-defined. Therefore, although a standard may include references to logarithms or trigonometry, those functions should not be included in coursework for Model Mathematics II; they will be addressed in Model Mathematics III.

For the high school Model Mathematics II course,[83] instructional time should focus on five critical areas: (1) extend the laws of exponents to rational exponents; (2) compare key characteristics of quadratic functions with those of linear and exponential functions; (3) create and solve equations and inequalities involving linear, exponential, and quadratic expressions; (4) extend work with probability; and (5) establish criteria for similarity of triangles based on dilations and proportional reasoning.

(1) Students extend the laws of exponents to rational exponents and explore distinctions between rational and irrational numbers by considering their decimal representations. Students learn that when quadratic equations do not have real solutions, the number system must be extended so that solutions exist, analogous to the way in which extending the whole numbers to the negative numbers allows $x + 1 = 0$ to have a solution. Students explore relationships between number systems: whole numbers, integers, rational numbers, real numbers, and complex numbers. The guiding principle is that equations with no solutions in one number system may have solutions in a larger number system.

(2) Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from among these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. When quadratic equations do not have real solutions, students learn that that the graph of the related quadratic function does not cross the horizontal axis. They expand their experience with functions to include more specialized functions—absolute value, step, and those that are piecewise-defined.

(3) Students begin by focusing on the structure of expressions, rewriting expressions to clarify and reveal aspects of the relationship they represent. They create and solve equations, inequalities, and systems of equations involving exponential and quadratic expressions.

(4) Building on probability concepts that began in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions.

[83] Adapted from the *Common Core State Standards for Mathematics* and *Appendix A: Designing High School Courses based on the Common Core State Standards for Mathematics.*

(5) Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. Students develop facility with geometric proof. They use what they know about congruence and similarity to prove theorems involving lines, angles, triangles, and other polygons. They explore a variety of formats for writing proofs.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Overview

Number and Quantity

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

The Complex Number Systems

- Perform arithmetic operations with complex numbers.
- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Solve equations and inequalities in one variable.
- Solve systems of equations.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic and exponential models and solve problems.

Trigonometric Functions

- Prove and apply trigonometric identities.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Geometry

Congruence

- Prove geometric theorems.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.

Statistics and Probability

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

MODEL INTEGRATED PATHWAY: Model Mathematics II

Content Standards

Number and Quantity

The Real Number System N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.[84]

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.[85]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.*
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions.[86]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

Write expressions in equivalent forms to solve problems.[87]

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

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[84] Limit Mathematics II to i^2 as highest power of i.
[85] Limit Mathematics II to quadratic equations with real coefficients.
[86] Expand to include quadratics and exponential expressions.
★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
[87] Expand to include quadratics and exponential expressions.

c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials.[88]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Creating Equations — A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.[89]* ★

Reasoning with Equations and Inequalities — A-REI

Solve equations and inequalities in one variable.[90]

4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
   MA.4.c. Demonstrate an understanding of the equivalence of factoring, completing the square, or using the quadratic formula to solve quadratic equations.

Solve systems of equations.[91]

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.*

Functions

Interpreting Functions — F-IF

Interpret functions that arise in applications in terms of the context.[92]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★

[88] Focus on adding and multiplying polynomial expressions; factor expressions to identify and collect like terms; and apply the distributive property.
★ indicates Modeling standard.
[89] Include formulas involving quadratic terms.
[90] Limit to quadratic equations with real coefficients.
[91] Expand to include linear/quadratic systems.
[92] Expand to include quadratic functions.

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations.[93]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, and $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.*
   MA.8.c. Translate among different representations of functions and relations: graphs, equations, point sets, and tables.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

MA.10. Given algebraic, numeric and/or graphical representations of functions, recognize the function as polynomial, rational, logarithmic, exponential, or trigonometric.

Building Functions F-BF

Build a function that models a relationship between two quantities.[94]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
   c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.* ★

Build new functions from existing functions.[95]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x - 1)$ for $x \neq 1$.*

★ indicates Modeling standard.
[93] Limit Mathematics II to linear, exponential, quadratic, piecewise-defined, and absolute value functions.
[94] Expand to include quadratic and exponential functions.
(+) indicates standard beyond College and Career Ready.
[95] Expand to include quadratic and absolute value functions.

Linear, Quadratic, and Exponential Models — F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Trigonometric Functions — F-TF

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant.

Geometry

Congruence — G-CO

Prove geometric theorems.[96]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*
    - MA.11.a. Prove theorems about polygons. *Theorems include: measures of interior and exterior angles, properties of inscribed polygons.*

Similarity, Right Triangles, and Trigonometry — G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
    a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
    b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

★ indicates Modeling standard.

[96] Focus on validity underlying reasoning and use a variety of ways of writing proofs.

Prove theorems involving similarity.[97]

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
   MA.3.a. Derive the formula for the relationship between the number of sides and sums of the interior and sums of the exterior angles of polygons and apply to the solutions of mathematical and contextual problems.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles.

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.[98]

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,* $\sqrt{3}$*) lies on the circle centered at the origin and containing the point (0, 2).*[99]
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

[97] Focus on validity underlying reasoning and use a variety of ways of writing proofs.
★ indicates a modeling standard.
[98] Limit Mathematics II use of radian to unit of measure.
[99] Include simple circle theorems.

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★★

Statistics and Probability

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data.[100]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

★ indicates Modeling standard.

100 Link to data from simulations and/or experiments.

Using Probability to Make Decisions **S-MD**

Use probability to evaluate outcomes of decisions.[101]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).[102] ★

[101] Introductory only; apply counting rules.
★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
[102] Replacing the hockey goalie with an extra skater.

Introduction

It is in the Model Mathematics III course that students integrate and apply the mathematics they have learned from their earlier courses. This course is comprised of standards selected from the high school **conceptual categories**, which were written to encompass the scope of content and skills to be addressed throughout grades 9–12 rather than through any single course. Therefore, the complete standard is presented in the model course, with clarifying footnotes as needed to limit the scope of the standard and indicate what is appropriate for study in this particular course. Standards that were limited in Model Mathematics I and Model Mathematics II no longer have those restrictions in Model Mathematics III.

For the high school Model Mathematics III course,[103] instructional time should focus on four critical areas: (1) apply methods from probability and statistics to draw inferences and conclusions from data; (2) expand understanding of functions to include polynomial, rational, and radical functions;[104] (3) expand right triangle trigonometry to include general triangles; and (4) consolidate functions and geometry to create models and solve contextual problems.

(1) Students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data— including sample surveys, experiments, and simulations—and the roles that randomness and careful design play in the conclusions that can be drawn.

(2) The structural similarities between the system of polynomials and the system of integers are developed. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. Rational numbers extend the arithmetic of integers by allowing division by all numbers except zero. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of the Model Mathematics III course is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. This critical area also includes exploration of the Fundamental Theorem of Algebra.

(3) Students derive the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

(4) Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying function. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as "the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions" is

[103] Adapted from the *Common Core State Standards for Mathematics* and *Appendix A: Designing High School Courses based on the Common Core State Standards for Mathematics.*

[104] In this course, rational functions are limited to those whose numerators are of degree at most 1 and denominators are of degree at most 2; radical functions are limited to square roots or cube roots of at most quadratic polynomials.

at the heart of this Model Mathematics III course. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Overview

Number and Quantity

The Complex Number System

- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Geometric Measurement and Dimension

- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

Content Standards

Number and Quantity

The Complex Number System — N-CN

Use complex numbers in polynomial identities and equations.[105]

8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.[106]

Algebra

Seeing Structure in Expressions[107] — A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
   MA.1.a. Divide polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[108]

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[105] Limit Mathematics III to polynomials with real coefficients.
[106] Expand to include higher-degree polynomials.
[107] Expand to polynomial and rational expressions.
★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
[108] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

Rewrite rational expressions.[109]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations — A-CED

Create equations that describe numbers or relationships.[110]

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.* ★

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning.

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically.

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Functions

Interpreting Functions — F-IF

Interpret functions that arise in applications in terms of the context.[111]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

[109] Focus on linear and quadratic denominators.
[110] Expand to include simple root functions.
★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
[111] Emphasize the selection of appropriate function model; expand to include rational, square, and cube functions.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations.[112]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   - c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   - a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   - b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, and $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.*
   - MA.8.c. Translate among different representations of functions and relations: graphs, equations, point sets, and tables.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

Build a function that models a relationship between two quantities.

1. Write a function that describes a relationship between two quantities. ★
   - b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions.[113]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   - a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x - 1)$ for $x \neq 1$.*

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.[114]

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★

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112 Expand to include rational and radical functions; focus on using key features to guide selection of appropriate type of function model.

★ indicates Modeling standard.

(+) indicates standard beyond College and Career Ready.

113 Expand to include simple radical, rational, and exponential functions; emphasize common effect of each transformation across function types.

114 Only include logarithms as solutions of exponential functions.

Trigonometric Functions **F-TF**

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Geometry

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Geometric Measurement and Dimension G-GMD

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

MA.4. Use dimensional analysis for unit conversions to confirm that expressions and equations make sense. ★

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions.

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).[115] ★

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
[115] Replacing the hockey goalie with an extra skater.

Introduction

Precalculus combines the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus, and strengthens students' conceptual understanding of problems and mathematical reasoning in solving problems. Facility with these topics is especially important for students intending to study calculus, physics, and other sciences, and/or engineering in college. Because the standards for this course are (+) standards, students selecting this Model Precalculus course should have met the college and career ready standards.

For the high school Model Precalculus course, instructional time should focus on four critical areas: (1) extend work with complex numbers; (2) expand understanding of logarithms and exponential functions; (3) use characteristics of polynomial and rational functions to sketch graphs of those functions; and (4) perform operations with vectors.

(1) Students continue their work with complex numbers. They perform arithmetic operations with complex numbers and represent them and the operations on the complex plane. Students investigate and identify the characteristics of the graphs of polar equations, using graphing tools. This includes classification of polar equations, the effects of changes in the parameters in polar equations, conversion of complex numbers from rectangular form to polar form and vice versa, and the intersection of the graphs of polar equations.

(2) Students expand their understanding of functions to include logarithmic and trigonometric functions. They investigate and identify the characteristics of exponential and logarithmic functions in order to graph these functions and solve equations and practical problems. This includes the role of *e*, natural and common logarithms, laws of exponents and logarithms, and the solutions of logarithmic and exponential equations. Students model periodic phenomena with trigonometric functions and prove trigonometric identities. Other trigonometric topics include reviewing unit circle trigonometry, proving trigonometric identities, solving trigonometric equations, and graphing trigonometric functions.

(3) Students investigate and identify the characteristics of polynomial and rational functions and use these to sketch the graphs of the functions. They determine zeros, upper and lower bounds, *y*-intercepts, symmetry, asymptotes, intervals for which the function is increasing or decreasing, and maximum or minimum points. Students translate between the geometric description and equation of conic sections. They deepen their understanding of the Fundamental Theorem of Algebra.

(4) Students perform operations with vectors in the coordinate plane and solve practical problems using vectors. This includes the following topics: operations of addition, subtraction, scalar multiplication, and inner (dot) product; norm of a vector; unit vector; graphing; properties; simple proofs; complex numbers (as vectors); and perpendicular components.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

MODEL ADVANCED COURSE: Model Precalculus

Overview

Number and Quantity

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.
- Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

Algebra

Arithmetic with Polynomials and Rational Expressions

- Use polynomial identities to solve problems
- Rewrite rational expressions.

Reasoning with Equations and Inequalities

- Solve systems of equations.

Functions

Interpreting Functions

- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Circles

- Understand and apply theorems about circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.
- Visualize relationships between two-dimensional and three-dimensional objects.

Content Standards

Number and Quantity

The Complex Number System — N-CN

Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $\left(-1+\sqrt{3}i\right)^3 = 8$ *because* $\left(-1+\sqrt{3i}\right)$ *has modulus 2 and argument 120°.*
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations.

8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x – 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities — N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.
   a. (+) Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. (+) Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. (+) Understand vector subtraction $\boldsymbol{v} – \boldsymbol{w}$ as $\boldsymbol{v} + (–\boldsymbol{w})$, where $–\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
   a. (+) Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
   b. (+) Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

(+) indicates standard beyond College and Career Ready.

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Arithmetic with Polynomials and Rational Expressions — A-APR

Use polynomial identities to solve problems.

5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[116]

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Reasoning with Equations and Inequalities — A-REI

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Functions

Interpreting Functions — F-IF

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★

[116] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.
(+) indicates standard beyond College and Career Ready.
★ indicates Modeling standard.

Building Functions — F-BF

Build a function that models a relationship between two quantities.

1. Write a function that describes a relationship between two quantities. ★
   c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.* ★

Build new functions from existing functions.

4. Find inverse functions.
   b. (+) Verify by composition that one function is the inverse of another.
   c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
   d. (+) Produce an invertible function from a non-invertible function by restricting the domain.
5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Trigonometric Functions — F-TF

Extend the domain of trigonometric functions using the unit circle.

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

Prove and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Geometry

Similarity, Right Triangles, and Trigonometry — G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = ½ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles — G-C

Understand and apply theorems about circles.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
   MA.3.a. (+) Use equations and graphs of conic sections to model real-world problems. ★

Geometric Measurement and Dimension — G-GMD

Explain volume formulas and use them to solve problems.

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.

Introduction

Because the standards for this course are (+) standards, students selecting this Model Advanced Quantitative Reasoning course should have met the college and career ready standards.

The high school Model Advanced Quantitative Reasoning course is designed as a mathematics course alternative to precalculus. Through this course, students are encouraged to continue their study of mathematical ideas in the context of real-world problems and decision-making through the analysis of information, modeling change, and mathematical relationships.

For the high school Model Advanced Quantitative Reasoning course, instructional time should focus on three critical areas: (1) critique quantitative data; (2) investigate and apply various mathematical models; and (3) explore and apply concepts of vectors and matrices to model and solve real-world problems.

(1) Students learn to become critical consumers of the quantitative data that surround them every day, knowledgeable decision-makers who use logical reasoning, and mathematical thinkers who can use their quantitative skills to solve problems related to a wide range of situations. They link classroom mathematics and statistics to everyday life, work, and decision-making, using mathematical modeling. They choose and use appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions.

(2) Through the investigation of mathematical models from real-world situations, students strengthen conceptual understandings in mathematics and further develop connections between algebra and geometry. Students use geometry to model real-world problems and solutions. They use the language and symbols of mathematics in representations and communication.

(3) Students explore linear algebra concepts of matrices and vectors. They use vectors to model physical relationships to define and solve real-world problems. Students draw, name, label, and describe vectors, perform operations with vectors, and relate these components to vector magnitude and direction. They use matrices in relationship to vectors and to solve problems.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

MODEL ADVANCED COURSE:

Model Advanced Quantitative Reasoning

Overview

Number and Quantity

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on matrices and use matrices in applications.

Algebra

Arithmetic with Polynomials and Rational Expressions

- Use polynomials identities to solve problems.

Reasoning with Equations and Inequalities

- Solve systems of equations.

Functions

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Circles

- Understand and apply theorems about circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.
- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Interpret linear models.

Making Inferences and Justifying Conclusions

- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Conditional Probability and the Rules of Probability

- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Calculate expected values and use them to solve problems.
- Use probability to evaluate outcomes of decisions.

Content Standards

Number and Quantity

Vector and Matrix Quantities — N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Arithmetic with Polynomials and Rational Expressions — A-APR

Use polynomial identities to solve problems.

5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[117]

Reasoning with Equations and Inequalities — A-REI

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Functions

Trigonometric Functions — F-TF

Extend the domain of trigonometric functions using the unit circle.

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

[117] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

(+) indicates standard beyond College and Career Ready.

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

Prove[118] and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Geometry

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

Understand and apply theorems about circles.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
   MA.3.a. (+) Use equations and graphs of conic sections to model real-world problems. ★

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-MG

Apply geometric concepts in modeling situations.

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

MA.4. Use dimensional analysis for unit conversions to confirm that expressions and equations make sense. ★

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Interpret linear models.

9. Distinguish between correlation and causation. ★

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
118 Advanced Quantitative Reasoning should accept informal proof and focus on the underlying reasoning, and use the theorems to solve problems.

Making Inferences and Justifying Conclusions S-IC

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Conditional Probability and the Rules of Probability S-CP

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Using Probability to Make Decisions S-MD

Calculate expected values and use them to solve problems.

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. ★
2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. ★
3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.* ★
4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?* ★

Use probability to evaluate outcomes of decisions.

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. ★
   a. (+) Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* ★
   b. (+) Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.* ★
6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).[119]

★ indicates Modeling standard.
(+) indicates standard beyond College and Career Ready.
119 Replacing the hockey goalie with an extra skater.

APPLICATION OF
COMMON CORE STATE STANDARDS FOR
English Language Learners
and
Students with Disabilities

Applications of Common Core State Standards for: English Language Learners

[120]The National Governors Association Center for Best Practices and the Council of Chief State School Officers strongly believe that all students should be held to the same high expectations outlined in the Common Core State Standards. This includes students who are English language learners (ELLs). However, these students may require additional time, appropriate instructional support, and aligned assessments as they acquire both English language proficiency and content area knowledge.

ELLs are a heterogeneous group with differences in ethnic background, first language, socioeconomic status, quality of prior schooling, and levels of English language proficiency. Effectively educating these students requires diagnosing each student instructionally, adjusting instruction accordingly, and closely monitoring student progress. For example, ELLs who are literate in a first language that shares cognates with English can apply first-language vocabulary knowledge when reading in English; likewise ELLs with high levels of schooling can often bring to bear conceptual knowledge developed in their first language when reading in English. However, ELLs with limited or interrupted schooling will need to acquire background knowledge prerequisite to educational tasks at hand. Additionally, the development of native-like proficiency in English takes many years and will not be achieved by all ELLs especially if they start schooling in the US in the later grades. Teachers should recognize that it is possible to achieve the standards for reading and literature, writing and research, language development, and speaking and listening without manifesting native-like control of conventions and vocabulary.

English Language Arts

The Common Core State Standards for English language arts (ELA) articulate rigorous grade-level expectations in the areas of speaking, listening, reading, and writing to prepare all students to be college and career ready, including English language learners. Second-language learners also will benefit from instruction about how to negotiate situations outside of those settings so they are able to participate on equal footing with native speakers in all aspects of social, economic, and civic endeavors.

ELLs bring with them many resources that enhance their education and can serve as resources for schools and society. Many ELLs have first language and literacy knowledge and skills that boost their acquisition of language and literacy in a second language; additionally, they bring an array of talents and cultural practices and perspectives that enrich our schools and society. Teachers must build on this enormous reservoir of talent and provide those students who need it with additional time and appropriate instructional support. This includes language proficiency standards that teachers can use in conjunction with the ELA standards to assist ELLs in becoming proficient and literate in English. To help ELLs meet high academic standards in language arts it is essential that they have access to:

- Teachers and personnel at the school and district levels who are well prepared and qualified to support ELLs while taking advantage of the many strengths and skills they bring to the classroom;
- Literacy-rich school environments where students are immersed in a variety of language experiences;
- Instruction that develops foundational skills in English and enables ELLs to participate fully in grade-level coursework;
- Coursework that prepares ELLs for postsecondary education or the workplace, yet is made comprehensible for students learning content in a second language (through specific pedagogical techniques and additional resources);
- Opportunities for classroom discourse and interaction that are well-designed to enable ELLs to develop communicative strengths in language arts;
- Ongoing assessment and feedback to guide learning; and
- Speakers of English who know the language well enough to provide ELLs with models and support.

Mathematics

ELL students are capable of participating in mathematical discussions as they learn English. Mathematics instruction for ELL students should draw on multiple resources and modes available in classrooms—such as objects, drawings, inscriptions, and gestures—as well as home languages and mathematical experiences outside of school. Mathematics instruction for ELLs should address mathematical discourse and academic language. This instruction involves much more than vocabulary lessons. Language is a resource for learning mathematics; it is not only a tool for communicating, but also a tool for thinking and reasoning mathematically. All languages and language varieties (e.g., different dialects, home or

[120] http://www.corestandards.org/the-standards

everyday ways of talking, vernacular, slang) provide resources for mathematical thinking, reasoning, and communicating.

Regular and active participation in the classroom—not only reading and listening but also discussing, explaining, writing, representing, and presenting—is critical to the success of ELLs in mathematics. Research has shown that ELLs can produce explanations, presentations, etc., and participate in classroom discussions *as they are learning English*.

ELLs, like English-speaking students, require regular access to teaching practices that are most effective for improving student achievement. Mathematical tasks should be kept at high cognitive demand, teachers and students should attend explicitly to concepts; and students should wrestle with important mathematics.

Overall, research suggests that:

- Language switching can be swift, highly automatic, and facilitate rather than inhibit solving word problems in the second language, as long as the student's language proficiency is sufficient for understanding the text of the word problem;
- Instruction should ensure that students understand the text of word problems before they attempt to solve them;
- Instruction should include a focus on "mathematical discourse" and "academic language" because these are important for ELLs. Although it is critical that students who are learning English have opportunities to communicate mathematically, this is not primarily a matter of learning vocabulary. Students learn to participate in mathematical reasoning, not by learning vocabulary, but by making conjectures, presenting explanations, and/or constructing arguments; and
- While vocabulary instruction is important, it is not sufficient for supporting mathematical communication. Furthermore, vocabulary drill and practice are not the most effective instructional practices for learning vocabulary. Research has demonstrated that vocabulary learning occurs most successfully through instructional environments that are language-rich, actively involve students in using language, require that students both understand spoken or written words and also express that understanding orally and in writing, and require students to use words in multiple ways over extended periods of time. To develop written and oral communication skills, students need to participate in negotiating meaning for mathematical situations and in mathematical practices that require output from students.

Application to Students with Disabilities

[121]The Common Core State Standards articulate rigorous grade-level expectations in the areas of mathematics and English language arts. These standards identify the knowledge and skills students need in order to be successful in college and careers.

Students with disabilities—students eligible under the Individuals with Disabilities Education Act (IDEA)—must be challenged to excel within the general curriculum and be prepared for success in their post-school lives, including college and/or careers. These common standards provide an historic opportunity to improve access to rigorous academic content standards for students with disabilities. The continued development of understanding about research-based instructional practices and a focus on their effective implementation will help improve access to mathematics and English language arts (ELA) standards for all students, including those with disabilities.

Students with disabilities are a heterogeneous group with one common characteristic: the presence of disabling conditions that significantly hinder their abilities to benefit from general education (IDEA 34 CFR §300.39, 2004). Therefore, *how* these high standards are taught and assessed is of the utmost importance in reaching this diverse group of students.

In order for students with disabilities to meet high academic standards and to fully demonstrate their conceptual and procedural knowledge and skills in mathematics, reading, writing, speaking and listening (English language arts), their instruction must incorporate supports and accommodations, including:

- Supports and related services designed to meet the unique needs of these students and to enable their access to the general education curriculum (IDEA 34 CFR §300.34, 2004).
- An Individualized Education Program (IEP)[122] which includes annual goals aligned with and chosen to facilitate their attainment of grade-level academic standards.
- Teachers and specialized instructional support personnel who are prepared and qualified to deliver high-quality, evidence-based, individualized instruction and support services.

Promoting a culture of high expectations for all students is a fundamental goal of the Common Core State Standards. In order to participate with success in the general curriculum, students with disabilities, as appropriate, may be provided additional supports and services, such as:

- Instructional supports for learning based on the principles of Universal Design for Learning (UDL),[123]– which foster student engagement by presenting information in multiple ways and allowing for diverse avenues of action and expression.
- Instructional accommodations (Thompson, Morse, Sharpe & Hall, 2005) – changes in materials or procedures which do not change the standards but allow students to learn within the framework of the Common Core.
- Assistive technology devices and services to ensure access to the general education curriculum and the Common Core State Standards.

Some students with the most significant cognitive disabilities will require substantial supports and accommodations to have meaningful access to certain standards in both instruction and assessment, based on their communication and academic needs. These supports and accommodations should ensure that students receive access to multiple means of learning and opportunities to demonstrate knowledge, but at the same time retain the rigor and high expectations of the Common Core State Standards.

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[121] http://www.corestandards.org/the-standards

[122] According to IDEA, an IEP includes appropriate accommodations that are necessary to measure the individual achievement and functional performance of a child.

[123] UDL is defined as "a scientifically valid framework for guiding educational practice that (a) provides flexibility in the ways information is presented, in the ways students respond or demonstrate knowledge and skills, and in the ways students are engaged; and (b) reduces barriers in instruction, provides appropriate accommodations, supports, and changes, and maintains high achievement expectations for all students including students with disabilities and students who are limited English proficient" by Higher Education Opportunity Act (PL 110–135).

Students with Disabilities

References

Individuals with Disabilities Education Act (IDEA), 34 CFR §300.34 (a). (2004).
Individuals with Disabilities Education Act (IDEA), 34 CFR §300.39 (b)(3). (2004).
Thompson, Sandra J., Amanda B. Morse, Michael Sharpe, and Sharon Hall. "Accommodations Manual: How to Select, Administer and Evaluate Use of Accommodations and Assessment for Students with Disabilities," 2nd Edition. Council for Chief State School Officers, 2005 http://www.ccsso.org/content/pdfs/AccommodationsManual.pdf. (Accessed January, 29, 2010).

GLOSSARY:
Mathematical Terms, Tables, and Illustrations

Glossary: Mathematical Terms, Tables, and Illustrations

This glossary contains those terms found and defined in the* Common Core State Standards for Mathematics*, as well as selected additional terms.

Glossary Sources

(DPI) http://dpi.wi.gov/standards/mathglos.html
(H) http://www.hbschool.com/glossary/math2/
(M) http://www.merriam-webster.com/
(MW) http://www.mathwords.com
(NCTM) http://www.nctm.org

AA similarity. Angle-angle similarity. When two triangles have corresponding angles that are congruent, the triangles are similar. (MW)

ASA congruence. Angle-side-angle congruence. When two triangles have corresponding angles and sides that are congruent, the triangles themselves are congruent. (MW)

Absolute value. A nonnegative number equal in numerical value to a given real number. (MW)

Addition and subtraction within 5, 10, 20, 100, or 1000. Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0–5, 0–10, 0–20, or 0–100, respectively. *Example: 8 + 2 = 10 is an addition within 10, 14 – 5 = 9 is a subtraction within 20, and 55 – 18 = 37 is a subtraction within 100.*

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. *Example:* $3/4$ *and* $–3/4$ *are additive inverses of one another because* $3/4 + (–3/4) = (–3/4) + 3/4 = 0$.

Algorithm. A finite set of steps for completing a procedure, e.g., long division. (H)

Analog. Having to do with data represented by continuous variables, e.g., a clock with hour, minute, and second hands. (M)

Analytic geometry. The branch of mathematics that uses functions and relations to study geometric phenomena, e.g., the description of ellipses and other conic sections in the coordinate plane by quadratic equations.

Argument of a complex number. The angle describing the direction of a complex number on the complex plane. The argument is measured in radians as an angle in standard position. For a complex number in polar form $r(\cos \theta + i \sin \theta)$, the argument is θ. (MW)

Associative property of addition. See Table 3 in this Glossary.

Associative property of multiplication. See Table 3 in this Glossary.

Assumption. A fact or statement (as a proposition, axiom, postulate, or notion) taken for granted. (M)

Attribute. A common feature of a set of figures.

Benchmark fraction. A common fraction against which other fractions can be measured, such as ½.

Binomial Theorem. A method for distributing powers of binomials. (MW)

Bivariate data. Pairs of linked numerical observations. *Example: a list of heights and weights for each player on a football team.*

Box plot. A graphic method that shows the distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data. (DPI)

Calculus. The mathematics of change and motion. The main concepts of calculus are limits, instantaneous rates of change, and areas enclosed by curves.

Capacity. The maximum amount or number that can be contained or accommodated, e.g., a jug with a one-gallon *capacity;* the auditorium was filled to *capacity.*

Cardinal number. A number (as 1, 5, 15) that is used in simple counting and that indicates how many elements there are in a set.

Glossary: Mathematical Terms, Tables, and Illustrations

Cartesian plane. A coordinate plane with perpendicular coordinate axes.

Cavalieri's Principle. A method, with formula given below, of finding the volume of any solid for which cross-sections by parallel planes have equal areas. This includes, but is not limited to, cylinders and prisms. Formula: Volume = Bh, where B is the area of a cross-section and h is the height of the solid. (MW)

Coefficient. Any of the factors of a product considered in relation to a specific factor. (W)

Commutative property. See Table 3 in this Glossary.

Complex fraction. A fraction A/B where A and/or B are fractions (B nonzero).

Complex number. A number that can be written as the sum or difference of a real number and an imaginary number. See Illustration 1 in this glossary. (MW)

Complex plane. The coordinate plane used to graph complex numbers. (MW)

Compose numbers. a) Given pairs, triples, etc. of numbers, identify sums or products that can be computed; b) Each place in the base ten place value is composed of ten units of the place to the left, i.e., one hundred is composed of ten bundles of ten, one ten is composed of ten ones, etc.

Compose shapes. Join geometric shapes without overlaps to form new shapes.

Composite number. A whole number that has more than two factors. (H)

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. *See also:* ***algorithm****;* ***computation strategy****.*

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *See also:* ***computation algorithm****.*

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

Conjugate. The result of writing sum of two terms as a difference, or vice versa. *For example, the conjugate of $x - 2$ is $x + 2$.* (MW)

Coordinate plane. A plane in which two *coordinate axes* are specified, i.e., two intersecting directed straight lines, usually perpendicular to each other, and usually called the x-axis and y-axis. Every point in a coordinate plane can be described uniquely by an ordered pair of numbers, the *coordinates* of the point with respect to the coordinate axes.

Cosine. A trigonometric function that for an acute angle is the ratio between a leg adjacent to the angle when the angle is considered part of a right triangle and the hypotenuse. (M)

Counting number. A number used in counting objects, i.e., a number from the set 1, 2, 3, 4, 5,…. See Illustration 1 in this Glossary.

Counting on. A strategy for finding the number of objects in a group without having to count every member of the group. *For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again; one can find the total by counting on—pointing to the top book and saying "eight," following this with "nine, ten, eleven. There are eleven books now."*

Decimal expansion. Writing a rational number as a decimal.

Decimal fraction. A fraction (as $0.25 = {}^{25}/_{100}$ or $0.025 = {}^{25}/_{1000}$) or mixed number (as $3.025 = 3\ {}^{25}/_{1000}$) in which the denominator is a power of ten, usually expressed by the use of the decimal point. (M)

Decimal number. Any real number expressed in base 10 notation, such as 2.673.

Decompose numbers. Given a number, identify pairs, triples, etc. of numbers that combine to form the given number using subtraction and division.

Decompose shapes. Given a geometric shape, identify geometric shapes that meet without overlap to form the given shape.

Digit. a) Any of the Arabic numerals 1 to 9 and usually the symbol 0; b) One of the elements that combine to form numbers in a system other than the decimal system.

Digital. Having to do with data that is represented in the form of numerical digits; providing a readout in numerical digits, e.g., a digital watch.

Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

Directrix. A fixed curve with which a generatrix maintains a given relationship in generating a geometric figure; specifically: a straight line the distance to which from any point in a conic section is in fixed ratio to the distance from the same point to a focus. (M)

Discrete mathematics. The branch of mathematics that includes combinatorics, recursion, Boolean algebra, set theory, and graph theory.

Dot plot. *See: **line plot***.

Expanded form. A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. *For example, $643 = 600 + 40 + 3$.*

Expected value. For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

Exponent. The number that indicates how many times the base is used as a factor, e.g., in $4^3 = 4 \times 4 \times 4 = 64$, the exponent is 3, indicating that 4 is repeated as a factor three times.

Exponential function. A function of the form $y = a \bullet b^x$ where $a > 0$ and either $0 < b < 1$ or $b > 1$. The variables do not have to be x and y. *For example, $A = 3.2 \bullet (1.02)^t$ is an exponential function.*

Expression. A mathematical phrase that combines operations, numbers, and/or variables (e.g., $3^2 \div a$). (H)

Fibonacci sequence. The sequence of numbers beginning with 1, 1, in which each number that follows is the sum of the previous two numbers, i.e., 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144….

First quartile. For a data set with median M, the first quartile is the median of the data values less than M. *Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6.*[124] *See also: **median, third quartile, interquartile range**.*

Fraction. A number expressible in the form a/b where a is a whole number and b is a positive whole number. (The word *fraction* in these standards always refers to a nonnegative number.) *See also: **rational number**.*

Function. A mathematical relation for which each element of the domain corresponds to exactly one element of the range. (MW)

Function notation. A notation that describes a function. For a function f, when x is a member of the domain, the symbol $f(x)$ denotes the corresponding member of the range (e.g., $f(x) = x + 3$).

Fundamental Theorem of Algebra. The theorem that establishes that, using complex numbers, all polynomials can be factored. A generalization of the theorem asserts that any polynomial of degree n has exactly n zeros, counting multiplicity. (MW)

Geometric sequence (progression). An ordered list of numbers that has a common ratio between consecutive terms, e.g., 2, 6, 18, 54…. (H)

Histogram. A type of bar graph used to display the distribution of measurement data across a continuous range.

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[124] Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Langford, E., "Quartiles in Elementary Statistics," *Journal of Statistics Education* Volume 14, Number 3 (2006).

Identity property of 0. See Table 3 in this Glossary.

Imaginary number. Complex numbers with no real terms, such as 5*i*. See Illustration 1 in this Glossary. (M)

Independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

Integer. All positive and negative whole numbers, including zero. (MW)

Interquartile range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is 15 – 6 = 9. *See also:* ***first quartile, third quartile***.

Inverse function. A function obtained by expressing the dependent variable of one function as the independent variable of another; that is the inverse of $y - f(x)$ is $x = f^{-1}(y)$. (NCTM)

Irrational number. A number that cannot be expressed as a quotient of two integers, e.g., $\sqrt{2}$. It can be shown that a number is irrational if and only if it cannot be written as a repeating or terminating decimal.

Law of Cosines. An equation relating the cosine of an interior angle and the lengths of the sides of a triangle. (MW)

Law of Sines. Equations relating the sines of the interior angles of a triangle and the corresponding opposite sides. (MW)

Line plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a dot plot. (DPI)

Linear association. Two variables have a linear association if a scatter plot of the data can be well-approximated by a line.

Linear equation. Any equation that can be written in the form $Ax + By + C = 0$ where A and B cannot both be 0. The graph of such an equation is a line.

Linear function. A mathematical function in which the variables appear only in the first degree, are multiplied by constants, and are combined only by addition and subtraction. *For example:* $f(s) = Ax + By + C$. (M)

Logarithm. The exponent that indicates the power to which a base number is raised to produce a given number. *For example, the logarithm of 100 to the base 10 is 2.* (M)

Logarithmic function. Any function in which an independent variable appears in the form of a logarithm; they are the inverse functions of exponential functions.

Matrix (pl. matrices). A rectangular array of numbers or variables.

Mean. A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.[125] *Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.*

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. *Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.*

Measure of variability. A determination of how much the performance of a group deviates from the mean or median, most frequently used measure is standard deviation.

Median. A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list; or the mean of the two central values, if the list contains an even number of values. *Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.*

[125] To be more precise, this defines the *arithmetic mean*.

Midline. In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values.

Model. A mathematical representation (e.g., number, graph, matrix, equation(s), geometric figure) for real-world or mathematical objects, properties, actions, or relationships. (DPI)

Modulus of a complex number. The distance between a complex number and the origin on the complex plane. The absolute value of $a + bi$ is written $|a + bi|$, and the formula for $|a + bi|$ is $\sqrt{a^2+b^2}$. For a complex number in polar form, $r(\cos\theta + i \sin \theta)$, the modulus is r. (MW)

Multiplication and division within 100. Multiplication or division of two whole numbers with whole number answers, and with product or dividend in the range 0–100. *Example: 72 ÷ 8 = 9*.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. *Example: $^3/_4$ and $^4/_3$ are multiplicative inverses of one another because $^3/_4 \times {}^4/_3 = {}^4/_3 \times {}^3/_4 = 1$.*

Network. a) A figure consisting of vertices and edges that shows how objects are connected, b) A collection of points (vertices), with certain connections (edges) between them.

Non-linear association. The relationship between two variables is nonlinear if a change in one is associated with a change in the other and depends on the value of the first; that is, if the change in the second is not simply proportional to the change in the first, independent of the value of the first variable.

Number line diagram. A diagram of the number line used to represent numbers and support reasoning about them. In a number line diagram for measurement quantities, the interval from 0 to 1 on the diagram represents the unit of measure for the quantity.

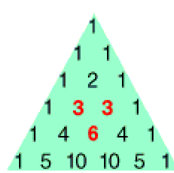
Numeral. A symbol or mark used to represent a number.

Order of Operations. Convention adopted to perform mathematical operations in a consistent order. 1. Perform all operations inside parentheses, brackets, and/or above and below a fraction bar in the order specified in steps 3 and 4; 2. Find the value of any powers or roots; 3. Multiply and divide from left to right; 4. Add and subtract from left to right. (NCTM)

Ordinal number. A number designating the place (as first, second, or third) occupied by an item in an ordered sequence. (M)

Partition. A process of dividing an object into parts.

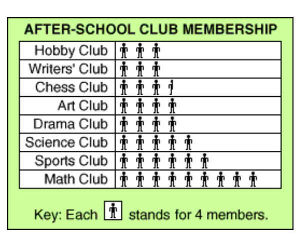
Pascal's triangle. A triangular arrangement of numbers in which each row starts and ends with 1, and each other number is the sum of the two numbers above it. (H)



Percent rate of change. A rate of change expressed as a percent. *Example: if a population grows from 50 to 55 in a year, it grows by 5/50 = 10% per year.*

Periodic phenomena. Naturally recurring events, for example, ocean tides, machine cycles.

Picture graph. A graph that uses pictures to show and compare information.



Glossary: Mathematical Terms, Tables, and Illustrations

Polar form. The polar coordinates of a complex number on the complex plane. The polar form of a complex number is written in any of the following forms: $r\cos\theta + r\,i\sin\theta$, $r(\cos\theta + i\sin\theta)$, or $r\,\mathrm{cis}\,\theta$. In any of these forms, r is called the modulus or absolute value. θ is called the argument. (MW)

Polynomial. The sum or difference of terms which have variables raised to positive integer powers and which have coefficients that may be real or complex. The following are all polynomials: $5x^3 - 2x^2 + x - 13$, $x^2y^3 + xy$, and $(1 + i)a^2 + ib^2$. (MW)

Polynomial function. Any function whose value is the solution of a polynomial.

Postulate. A statement accepted as true without proof.

Prime factorization. A number written as the product of all its prime factors. (H)

Prime number. A whole number greater than 1 whose only factors are 1 and itself.

Probability distribution. The set of possible values of a random variable with a probability assigned to each.

Properties of equality. See Table 4 in this Glossary.

Properties of inequality. See Table 5 in this Glossary.

Properties of operations. See Table 3 in this Glossary.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, testing for a medical condition).

Probability model. A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. *See also: **uniform probability model***.

Proof. A method of constructing a valid argument, using deductive reasoning.

Proportion. An equation that states that two ratios are equivalent, e.g., $^4/_8 = \frac{1}{2}$ or 4 : 8 = 1 : 2.

Pythagorean theorem. For any right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.

Quadratic equation. An equation that includes only second degree polynomials. Some examples are $y = 3x^2 - 5x^2 + 1$, $x^2 + 5xy + y^2 = 1$, and $1.6a^2 + 5.9a - 3.14 = 0$. (MW)

Quadratic expression. An expression that contains the square of the variable, but no higher power of it.

Quadratic function. A function that can be represented by an equation of the form $y = ax^2 + bx + c$, where a, b, and c are arbitrary, but fixed, numbers and a 0. The graph of this function is a parabola. (DPI)

Quadratic polynomial. A polynomial where the highest degree of any of its terms is 2.

Radical. The $\sqrt{\ }$ symbol, which is used to indicate square roots or nth roots. (MW)

Random sampling. A smaller group of people or objects chosen from a larger group or population by a process giving equal chance of selection to all possible people or objects. (H)

Random variable. An assignment of a numerical value to each outcome in a sample space. (M)

Ratio. A comparison of two numbers or quantities, e.g., 4 to 7 or 4 : 7 or $^4/_7$.

Rational expression. A quotient of two polynomials with a non-zero denominator.

Rational number. A number expressible in the form $^a/_b$ or $-\ ^a/_b$ for some fraction $^a/_b$. The rational numbers include the integers. See Illustration 1 in this Glossary.

Real number. A number from the set of numbers consisting of all rational and all irrational numbers. See Illustration 1 in this Glossary.

Rectangular array. An arrangement of mathematical elements into rows and columns.

Rectilinear figure. A polygon all angles of which are right angles.

Recursive pattern or sequence. A pattern or sequence wherein each successive term can be computed from some or all of the preceding terms by an algorithmic procedure.

Reflection. A type of transformation that flips points about a line, called the *line of reflection.* Taken together, the image and the pre-image have the line of reflection as a line of symmetry.

Relative frequency. The empirical counterpart of probability. If an event occurs N' times in N trials, its relative frequency is N'/N. (M)

Remainder Theorem. If $f(x)$ is a polynomial in x then the remainder on dividing $f(x)$ by $x - a$ is $f(a)$. (M)

Repeating decimal. A decimal in which, after a certain point, a particular digit or sequence of digits repeats itself indefinitely; the decimal form of a rational number. (M) *See also:* ***terminating decimal****.*

Rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Rotation. A type of transformation that turns a figure about a fixed point, called the *center of rotation*.

SAS congruence. (Side-angle-side congruence.) When two triangles have corresponding sides and the angles formed by those sides are congruent, the triangles are congruent. (MW)

SSS congruence. (Side-side-side congruence.) When two triangles have corresponding sides that are congruent, the triangles are congruent. (MW)

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

Scatter plot. A graph in the coordinate plane representing a set of bivariate data. *For example, the heights and weights of a group of people could be displayed on a scatter plot.* (DPI)

Scientific notation. A widely used floating-point system in which numbers are expressed as products consisting of a number between 1 and 10 multiplied by an appropriate power of 10, e.g., $562 = 5.62 \times 10^2$. (MW)

Sequence, progression. A set of elements ordered so that they can be labeled with consecutive positive integers starting with 1, e.g., 1, 3, 9, 27, 81. In this sequence, 1 is the *first term*, 3 is the *second term*, 9 is the *third term*, and so on.

Significant figures. (digits) A way of describing how precisely a number is written, particularly when the number is a measurement. (MW)

Similarity transformation. A rigid motion followed by a dilation.

Simultaneous equations. Two or more equations containing common variables. (MW)

Sine. The trigonometric function that for an acute angle is the ratio between the leg opposite the angle when the angle is considered part of a right triangle and the hypotenuse. (M)

Tangent. a) Meeting a curve or surface in a single point if a sufficiently small interval is considered. b) The trigonometric function that, for an acute angle, is the ratio between the leg opposite the angle and the leg adjacent to the angle when the angle is considered part of a right triangle. (MW)

Tape diagram. A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model.

Terminating decimal. A decimal is called terminating if its repeating digit is 0. A terminating decimal is the decimal form of a rational number. *See also:* ***repeating decimal***.

Third quartile. For a data set with median M, the third quartile is the median of the data values greater than M. *Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. See also:* ***median, first quartile, interquartile range****.*

Transformation. A prescription, or rule, that sets up a one-to-one correspondence between the points in a geometric object (the *pre-image*) and the points in another geometric object (the *image*). Reflections, rotations, translations, and dilations are particular examples of transformations.

Transitivity principle for indirect measurement. If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Translation. A type of transformation that moves every point in a graph or geometric figure by the same distance in the same direction without a change in orientation or size. (MW)

Trigonometric function. A function (as the sine, cosine, tangent, cotangent, secant, or cosecant) of an arc or angle most simply expressed in terms of the ratios of pairs of sides of a right-angled triangle. (M)

Trigonometry. The study of triangles, with emphasis on calculations involving the lengths of sides and the measure of angles. (MW)

Uniform probability model. A probability model which assigns equal probability to all outcomes. *See also: **probability model**.*

Unit fraction. A fraction with a numerator of 1, such as $^1/_3$ or $^1/_5$.

Valid. a) Well-grounded or justifiable; being at once relevant and meaningful, e.g., a *valid* theory; b) Logically correct. (MW)

Variable. A quantity that can change or that may take on different values. Refers to the letter or symbol representing such a quantity in an expression, equation, inequality, or matrix. (MW)

Vector. A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

Visual fraction model. A tape diagram, number line diagram, or area model.

Whole numbers. The numbers 0, 1, 2, 3, … . See Illustration 1 in this Glossary.

Tables and Illustrations
of Key Mathematical Properties, Rules, and Number Sets

TABLE 1. Common addition and subtraction situations.[126]

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3

	Total Unknown	Addend Unknown	Both Addends Unknown[127]
Put Together/ Take Apart[128]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[129]	(“How many more?” version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? (“How many fewer?” version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with “more”): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with “more”): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

[126] Adapted from Boxes 2–4 of *Mathematics Learning in Early Childhood*, National Research Council (2009, pp. 32–33).
[127] These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*.
[128] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.
[129] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

Glossary: Mathematical Terms, Tables, and Illustrations

TABLE 2. Common multiplication and division situations.[130]

	Unknown Product **$3 \times 6 = ?$**	**Group Size Unknown** ("How many in each group?" Division) **$3 \times ? = 18$ and $18 \div 3 = ?$**	**Number of Groups Unknown** ("How many groups?" Division) **$? \times 6 = 18$ and $18 \div 6 = ?$**
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[131] Area[132]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

[130] The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

[131] The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

[132] Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

TABLE 3. The properties of operations.

Here *a*, *b* and *c* stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$.
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$.
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

TABLE 4. The properties of equality.

Here *a*, *b*, and *c* stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.

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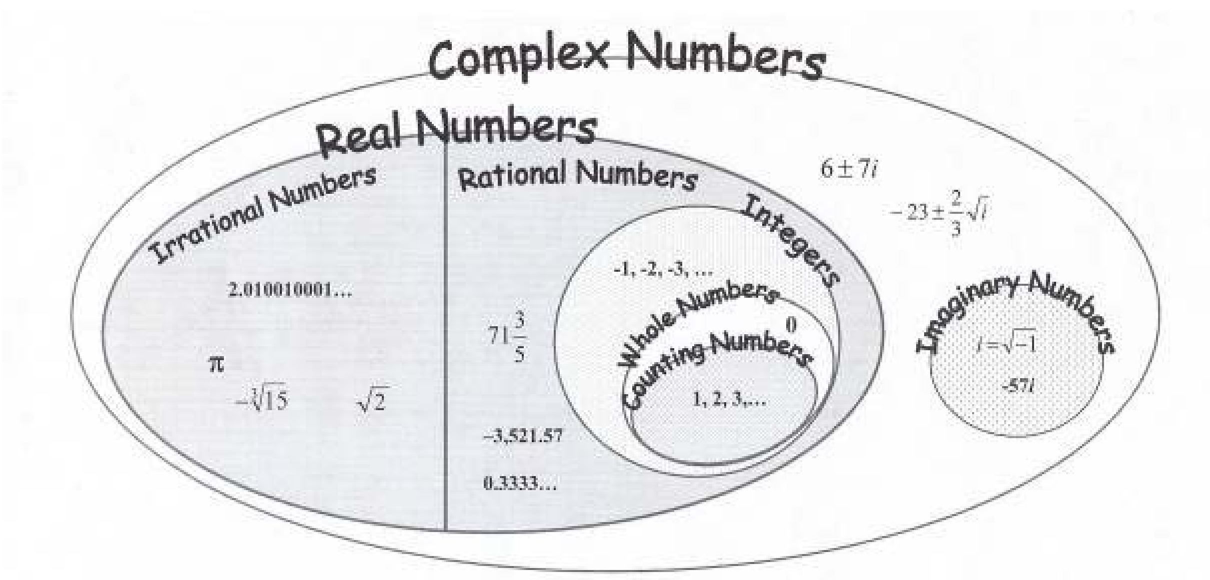
TABLE 5. The properties of inequality.

Here *a*, *b*, and *c* stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
If $a > b$ and $b > c$ then $a > c$.
If $a > b$, then $b < a$.
If $a > b$, then $-a < -b$.
If $a > b$, then $a \pm c > b \pm c$.
If $a > b$ and $c > 0$, then $a \times c > b \times c$.
If $a > b$ and $c < 0$, then $a \times c < b \times c$.
If $a > b$ and $c > 0$, then $a \div c > b \div c$.
If $a > b$ and $c < 0$, then $a \div c < b \div c$.

ILLUSTRATION 1. The Number System.

The Number System is comprised of number sets beginning with the Counting Numbers and culminating in the more complete Complex Numbers. The name of each set is written on the boundary of the set, indicating that each increasing oval encompasses the sets contained within. Note that the Real Number Set is comprised of two parts: Rational Numbers and Irrational Numbers.



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